Unit 4

n-order linear ODE

$$L(y) = P_0(t)\, y^{(n)}(t) + P_1(t)\, y^{(n-1)}(t) + \dots + P_n(t)\, y = g(t)$$

And the initial condition

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \dots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Uniqueness and existence thm:

If $P_i(t)$ and $g(t)$ are cts on open interval I containing t_0, $\exists!$ solution $y = \phi(t)$ satisfies above ODE and the initial condition

Consider the homogenous case,

$$L(y) = 0$$

Def: the wronkian of $L(y) = 0$ to solutions $y_1, y_2, \dots y_n$ is

$$W(y_1, y_2 \dots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Thm If p_i are cts on open I, and y_i are the solution of $L(y)=0$ and $W(y_1 \dots y_n)(t) \neq 0$ for at least 1 point, then y_i are linearly independent, and so the solution of $L(y)=0$ can be expressed as linear combination of y_i.

We consider the case p_i are constant.

let $L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$

Set up the characteristic equation:

$$a_0 r^n + a_1 r^{n-1} + \dots + a_n = 0$$

let $r_1 \dots r_n$ be the roots,

If some r_i are real,

first part of solution $= \sum A_i e^{r_i t}$

If some r_i are repeated n times,

second part of solution $= \sum_{j=1} B_j t^{j-1} e^{r_i t}$

If some $r_j = u_j + i v_j$ are complex

third part of solution $= \sum_{j=1} e^{u_j t}(A_i \cos v_j t + B_i \sin v_j t)$

then the solution is the sum of these 3 parts.

Consider the nonhomogenous case

$$L(y)=g(t),$$

To find the solution corresponding to nonhomogenous part, there are 2 methods,

Method 1, Undetermined coefficients,

exactly same as the second order case.

Method 2,

If y_i are the solution corresponding to the homogenous case (linearly independent),

then let

$$W_m(s)=\begin{vmatrix} y_1 & y_2 & \cdots & 0 & \cdots & y_n \\ y_1' & y_2' & & 0 & & y_n' \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & 1 & & y_n^{(n-1)} \end{vmatrix}$$

m-column

then $Y(t)=\sum_{m=1}^{n} y_m(t)\left(\int_{t_0}^{t}\frac{g(s)W_m(s)}{W(s)}\,ds\right)$

then the general will be $y=$ homogenous part $+\,Y(t)$.

Abel's identity:

$W(y_1 \ldots y_n)(t) = C e^{-\int p_1 dt}$ if $y_1 \ldots y_n$ are ~~linearly independent~~ solutions of

$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n(t) y = 0.$

Problem:

① $y^{(7)} - 8y^{(6)} + 16y^{(3)} = 0,$

Ans: $r^7 - 8r^6 + 16r^3 = r^3(r^4 - 8r^3 + 16)$

$= r^3(r^2-4)^2 = 0,$

$r = 0$ (repeated 3 times)

$r = \pm 2$ (repeated twice, each)

$\therefore \; y = C_1 t^2 + C_2 t + C_3 + C_4 e^{2t} + C_5 e^{-2t} +$
$t(C_6 e^{2t} + C_7 e^{-2t})$

②: $y^{(4)} - y = 3t + \cos t$

$r^4 - 1 = 0$

$r = \; , -1, 1, \pm i$

$y_c(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t,$

and let $Y(t) = A_0 + A_1 t + t(A_2\cos t + A_3 \sin t)$

Because $A_2\cos t + A_3\sin t$ may be linearly dependent with $C_3\cos t + C_4\sin t$.

So we have $-A_0 + A_1 t - 4A_3\cos t - 4A_2 = 3t + \cos t$

$A_0 = 0,\ A_1 = -3,\ A_2 = 0,\ A_3 = -\frac{1}{4},$

$y = C_1 e^{t} + C_2 e^{-t} + C_3\cos t + C_4 \sin t - 3t - \frac{t}{4}\sin t.$

(3) $y''' + y' = \tan t \quad 0 < t < \frac{\pi}{2}$

Ans: $r^3 + r = 0 \Rightarrow r = \pm i, 0,$

So $y_c(t) = A + (B_1\sin t + B_2\cos t) + (C_1\sin t + C_2\cos t)$

$= A + B\sin t + C\cos t$

$$W(s) = \begin{vmatrix} 1 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = \begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix}$$

$= -1$

$$W_1 = \begin{vmatrix} 0 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 1 & -\sin t & -\cos t \end{vmatrix} = -1 \quad W_2 = \begin{vmatrix} 1 & 0 & \cos t \\ 0 & 0 & -\sin t \\ 0 & 1 & -\cos t \end{vmatrix} = +\sin t.$$

$$W_3 = \begin{vmatrix} 1 & \sin t & 0 \\ 0 & \cos t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = \cos t.$$

$$Y(t) = \int \frac{\tan t \cdot 1}{1} + \cos t \int \frac{-\tan t \cos t}{1} + \sin t \int \frac{-\tan t \sin t}{1}$$

$$= -\ln \cos t + \cos^2 t - \sin t\,(\ln|\sec t + \tan t| - \sin t)$$

(4) $y^{(4)} + 2y^{(2)} + y = 3 + \cos 2t + t \sin 3t + e^{5t}$,

Ans $r^4 + 2r^2 + 1 = 0$

$r = \pm i$ (repeated)

So $y_c(t) = C_1 \sin t + C_2 \cos t + C_3 t \sin t + C_4 t \cos t$,

let $Y(t) = A + B \sin 2t + C \cos 2t +$

$(1 + Dt)(E \sin 3t + F \cos 3t) + G e^{5t}$

$\therefore\ y(t) = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t$

$+ 3 + \frac{1}{9}\cos 3t + \frac{3}{128}\cos 3t + \frac{t}{64}\sin 3t + \frac{e^{5t}}{576}.$